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COVER SHEET FOR TECHNICAL MEMORANDUM**TITLE-** Availability and Reliability of
Some Models of the RTCC**TM-** 67-1031-2**DATE-** December 29, 1967**FILING CASE NO(S)-** 103**AUTHOR(S)-** J. J. Rocchio**FILING SUBJECT(S)-** Reliability
(ASSIGNED BY AUTHOR(S)-Markov Model**ABSTRACT**

The Real Time Computer Complex (RTCC), a part of the Mission Control Center, Houston, is modeled as a set of identical elements (computers), each with a constant failure rate (λ) and a constant repair rate (μ). Application of the theory of finite Markov processes to this model allows computation of various parameters of system behavior as a function of the total number of system elements, the number of repair crews, and the system requirements. In addition, the solution of a set of simultaneous linear differential equations provides the reliability functions of the system.

For a system of five computers, each having a mean time to failure of 70 hours, and a mean time to repair of 2 hours, at least three repair crews, and a worst case system requirement to simultaneously support two missions, the model predicts:

1. An availability of greater than 99.99%,
2. A mean time to failure of 9667 hours, and
3. A reliability for 50 hours of .9951.


(NASA-CR-92918) AVAILABILITY AND
RELIABILITY OF SOME MODELS OF THE RTCC
(Bellcomm, Inc.) 40 p

N79-72900

Unclas
12632

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FF No. 602(C)	(PAGES)	(CODE)
	CR-92918	08
(NASA CR OR TMX OR AD NUMBER)		(CATEGORY)



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1100 Seventeenth Street, N.W. Washington, D. C. 20036

SUBJECT: Availability and Reliability of Some
Models of the RTCC - Case 103

DATE: December 29, 1967

FROM: J. J. Rocchio

TM- 67-1031-2

TECHNICAL MEMORANDUM

I. INTRODUCTION

The Real Time Computer Complex (RTCC), a part of the Mission Control Center, Houston (MCCH) consists of a number of digital computers and ancillary equipment organized to support manned space flight missions. In this memorandum some simple probabilistic models of this system are analyzed to provide parameters associated with system availability and reliability.

The models (described below) are based on the following assumptions:

1. The system consists of N identical machines which are continuously operating. Each machine is characterized by an exponential failure law with mean time between failures of $1/\lambda$, and an exponential repair law with mean time to repair of $1/\mu$. This corresponds to assuming constant failure and repair rates.
2. A single machine can support any given mission or mission phase. Two machines are allocated to support a critical mission phase, one in control and the other in dynamic standby. In this configuration we assume perfect error detection and instantaneous switching so that failures of the primary machine are not system failures. In addition, we assume that mission dedicated machines which fail can be instantaneously replaced (if additional machines are available) since in practice only about ten seconds are required to effect this. The worst case to be considered is the requirement to support two critical phases. In considering system availability, we will assume that a machine not in dynamic standby can take over mission support in a time governed by an exponential distribution with mean $1/\delta$ (a mathematical convenience since in reality this time is probably constant).

II. SYSTEM AVAILABILITY

One of the important characteristics of the system under consideration is the distribution of the number of machines which will be operating as a function of the failure and repair time statistics. We consider the system at any time to be in any one of the $N + 1$ states S_K , $K = 0, 1, \dots, N$, where K denotes the number of machines working; i.e., when the system is in state S_K , K machines are operating and $N-K$ have failed. Depending on the number of repair crews available, all or some fraction of the $N-K$ failed machines are in the process of being repaired.

The earlier assumptions of constant failure rate and constant repair rate allow the system to be modeled as a finite Markov process. Consider that the system is in state S_K . Then in a small period of time Δt it can:

1. remain in state S_K ;
2. go to state S_{K+1} if a machine is repaired; or
3. go to state S_{K-1} if a machine fails.

With Δt sufficiently small, these are the only possibilities since the probabilities of multiple failures or repairs in a time interval Δt are of the order of Δt^2 or higher and may therefore be neglected in a limiting process as $\Delta t \rightarrow 0$.

The behavior of the system may now be described by a probability transition matrix P , where p_{ij} is the probability that the system, in state i at time t , will go to state j at time $t + \Delta t$. Assuming N or more repair crews are available, for state S_K ($0 < K < N$) we have:

$$p_{K,K+1} = (N-K) \mu \Delta t$$

*The probability that the system is in state S_K at time t will be denoted $P(S_K, t)$.

i.e., the probability that a machine is repaired;

$$p_{K,K-1} = K\lambda\Delta t$$

i.e., the probability that a machine fails; and

$$p_{K,K} = 1 - [(N-K)\mu + K\lambda] \Delta t$$

i.e., the probability that neither a failure nor a repair occurs.
The transition matrix is completed by noting that:

$$p_{N,N} = 1 - N\lambda\Delta t$$

$$p_{N,N-1} = N\lambda\Delta t$$

and

$$p_{0,0} = 1 - N\mu\Delta t$$

$$p_{0,1} = N\mu\Delta t$$

Figure 1 illustrates the system model.

If we let the initial state of the system be specified by a probability vector $\underline{p}^{(0)}$, then it can be shown that the probability vector characterizing the system after n steps, $\underline{p}^{(n)}$ is given by:

$$\underline{p}^{(n)} = \underline{p}^{(0)} P^n .$$

It may also be shown that

$$\lim_{n \rightarrow \infty} P^n = W$$

where each row of W is a probability vector $\underline{\omega}$, which satisfies

$$\underline{\omega}P = \underline{\omega}$$

or

$$\underline{\omega} \cdot [P-I] = 0$$

The vector $\underline{\omega}$ is called the fixed point of the Markov process and the elements of $\underline{\omega}$ may be interpreted as the equilibrium probabilities of the states of the process.*

*See for example Finite Mathematics with Business Applications, J. G. Kemeny, et al., Prentice Hall, 1962.

Case 1

With N machines and N or more repair crews (the model of Figure 1), Appendix A shows that the equilibrium probabilities of the states of the system are:

$$\omega_K = \lim_{t \rightarrow \infty} P(S_K, t) = P(S_K) = \frac{\binom{N}{K} \rho^{N-K}}{(1+\rho)^N} \quad *$$

where $\rho = \lambda/\mu$.

System availability may be defined as the probability that the system is operating satisfactorily at any point in time. To relate system availability to the equilibrium probabilities we must define the system requirements in terms of the number of machines required. For example, to compute system availability assuming that at least J of N machines are required for satisfactory system operation, we get (in equilibrium)

$$P_A(J) = P(S_J) + P(S_{J+1}) \dots + P(S_N) = \sum_{K=J}^N \omega_K$$

i.e., $P_A(J)$ is just the probability that at least J machines are operating. Table 1 tabulates these probabilities for some cases of interest as a function of selected values of $\rho = \lambda/\mu$. (Note that statistics gathered on RTCC operation through September 1967 indicate that a failure rate (λ) of 1/70 failures per hour, and a repair rate (μ) of .5 repairs per hour are representative of RTCC computers; with these data, the ratio $\rho = \lambda/\mu$ is 1/35.

*A general solution for $P(S_K, t)$ may be found in Bellcomm Memorandum for File, "A Birth-Death Process Associated With A Redundant Repairable System" by G. M. Anderson, (to be released). It is shown there that $P(S_K, t)$ approaches $P(S_K)$, its asymptotic value with a limiting exponential time constant of $1/(\lambda+\mu)$. Thus, after a period of time equal at most to 3 or 4 times the mean time to repair ($1/\mu$), the system is essentially stationary with distribution ω_K , justifying the use of the latter here.

J	N=5			N=6		
	$\rho=1/30$	$\rho=1/35$	$\rho=1/40$	$\rho=1/30$	$\rho=1/35$	$\rho=1/40$
6				.821405	.844487	.862296
5	.848785	.868615	.883854	.985686	.989256	.991641
4	.990249	.992703	.994336	.999376	.999597	.999725
3	.999680	.999794	.999860	.999984	.999991	.999994
2	.999994	.999997	.999998	.999999	.999999	.999999

Probability of Having J Out of N Machines Operating

Table 1

Case 2

The effect of a limited number of repair crews on the behavior of the system may be easily accounted for by changing the probability transition matrix considered in Case 1. The general solution for N machines and M ($M < N$) repair crews derived in Appendix A yields:

$$\omega_K = \frac{\binom{N}{K} \rho^{N-K}}{\sigma}, \quad \text{for } N-M \leq K \leq N \quad ;$$

and

$$\omega_K = \frac{\binom{N}{M} \rho^{N-K} \frac{(N-M)!}{K! M^{N-K-M}}}{\sigma}, \quad \text{for } 0 \leq K < N-M ;$$

where σ is such that

$$\sum_{K=0}^N \omega_K = 1, \text{ namely}$$

$$\sigma = \sum_{K=N-M}^N \binom{N}{K} \rho^{N-K} + \sum_{K=0}^{N-M-1} \binom{N}{M} \rho^{N-K} \frac{(N-M)!}{K! M^{N-K-M}} .$$

Table 2 summarizes the probabilities of at least J of N machines working, $P_A(J, M)$, and shows the effect of a limited number (M) of repair crews.

N=5

J	M=1			M=3		
	$\rho=1/30$	$\rho=1/35$	$\rho=1/40$	$\rho=1/30$	$\rho=1/35$	$\rho=1/40$
5	.839444	.861574	.878359	.848783	.868614	.883853
4	.979351	.984656	.988154	.990247	.992702	.994335
3	.998006	.998723	.999134	.999678	.999793	.999859
2	.999871	.999929	.999957	.999992	.999996	.999997

N=6

J	M=1			M=3		
	$\rho=1/30$	$\rho=1/35$	$\rho=1/40$	$\rho=1/30$	$\rho=1/35$	$\rho=1/40$
6	.807589	.834048	.854136	.821400	.844485	.862295
5	.969107	.977028	.982257	.985680	.989253	.991639
4	.996027	.997453	.998272	.999370	.999594	.999723
3	.999616	.999788	.999873	.999979	.999988	.999993
2	.999975	.999988	.999993	.999999	.999999	.999999

Probability of Having J of N Machines Operating, Given M
($M < N$) Repair Crews

Table 2

Case 3

The previous model, which predicts the relative frequency of the various system states, does not reflect the RTCC requirement to simultaneously support two critical mission phases. This situation requires the use of a pair of machines on each phase, one in control and one in dynamic standby. Consider the configuration of the system when only three machines are operating. In this case, only one mission, call it mission A, can be supported by a dynamic standby machine. Thus, if a failure occurs in either of the machines supporting mission A, the system is still in a satisfactory state; but if the machine supporting mission B

fails, the system has failed. This may be modeled by splitting the state S_2 into two states, S_{21} and S_{22} . In S_{21} we assume that each machine is supporting a different mission so that this is a success state for the system, whereas S_{22} corresponds to the case where both operating machines are supporting the same mission and the system has therefore failed. Figure 2 shows the system model and transition probabilities.

The equilibrium probabilities in this case are shown in Appendix A to be the same as in the previous model for the states S_i , $i \neq 2$. In addition, it is shown that

$$\omega_{22} + \omega_{21} = \omega_2$$

which merely states that the total probability of having two machines operating is not changed by the splitting of state S_2 into two states.

It is further shown that

$$\omega_{22} = \frac{\lambda \omega_3}{2\lambda + \alpha \mu},$$

where

$$\alpha = \min [N-2, M],$$

with N and M as previously defined.

This result now allows computation of the RTCC system availability to simultaneously support two critical mission phases as

$$P_A = 1 - P(S_{22}) - P(S_{21}) - P(S_0) = 1 - [\omega_{22} + \omega_{21} + \omega_0]$$

This probability is tabulated below as a function of N , the number of machines and M , the number of repair crews for some cases of interest.

No. of Repair Crews	Total Number of Machines					
	N=5			N=6		
	$\rho=1/30$	$\rho=1/35$	$\rho=1/40$	$\rho=1/30$	$\rho=1/35$	$\rho=1/40$
M						
1	.999291	.999548	.999696	.999863	.999925	.999955
2	.999832	.999892	.999927	.999983	.999991	.999994
3	.999890	.999929	.999952	.999992	.999996	.999997
4	.999892	.999930	.999952	.999994	.999997	.999998
5	.999892	.999930	.999952	.999994	.999997	.999998
6	.999892	.999930	.999952	.999994	.999997	.999998

System Availability (P_A) for Two Critical Missions

Table 3

As a point of interpretation, note that with one repair crew, the likelihood of the system being down with this model is 5-6 times that of a system which requires any two machines working. With more than one repair crew, this ratio increases. Note also that the gain in system availability per added repair crew decreases markedly after the first increment.

Case 4

The model of Case 3 must be slightly refined to reflect realistically the operating characteristics of the RTCC. Consider the situation when the system is in state S_{22} . As previously defined, in this state two machines are up, but both are assigned to a single mission. Clearly, in this situation the system would be reconfigured, and one of the machines would be switched to support the other mission. During this reconfiguration time, the system is, of course, in a failed state. The modification of the system model to account for this refinement is shown in Figure 3. Assuming that the switch-over time is exponentially distributed with mean $1/\delta$ rather than a constant allows the following result:

$$\omega_{22} = \frac{\lambda \omega_3}{2\lambda + \alpha\mu + \delta}$$

as shown in Appendix A. Table 4 shows system availabilities in this case where it has been assumed that $1/\delta = 5$ min.

No. of Repair Crews M	Total Number of Machines	
	N=5	N=6
	$\rho=1/35$	$\rho=1/35$
1	.999913	.999985
2	.999983	.999998
3	.999988	.999998
4	.999989	.999999
5	.999989	.999999
6	.999989	.999999

System Availability for Two Critical Phases (P_A)

Table 4

Note that the likelihood of the system being down is reduced by a factor of at least 6 when switchover from state S_{22} to S_{21} is accounted for.

III. RELIABILITY

The previous results concern aspects of system availability or the long term steady-state probabilities of various system states. In this section, we will consider aspects connected with system reliability $R(t)$, where we define reliability as the probability that the system will function satisfactorily for a given period of time. Derivation of the system reliability function involves the solution of a set of simultaneous linear differential equations (see Appendix C). But a parameter of this function, the mean time to failure (MTF) defined by:

$$MTF = \int_0^{\infty} R(t)dt \quad ,$$

can be found by application of theory of Markov processes with absorbing barriers. In addition to the intrinsic value of the MTF as an evaluation index of the system design, the results of

Appendix C suggest that for values of λ and μ of interest here ($\lambda \ll \mu$), $R(t)$ may be closely approximated by:

$$R(t) = e^{-t/MTF}$$

A. Mean Time to System Failure

To compute the MTF, we change the system model by making those states which are associated with system failure absorbing barriers. This means that whenever the system enters such a state it remains there with probability 1, and is thus trapped. This device allows the computation of the mean time to enter all such states starting from some given initial state which is exactly the desired MTF or mean time to system failure. The system model in this case is shown in Figure 4.

The probability transition matrix for any Markov chain with n states and r absorbing barriers can be put into the following standard form:

$$P = \begin{bmatrix} I & O \\ R & Q \end{bmatrix}$$

where I is an r by r identity matrix, R is an $n-r$ by r matrix, and Q is an $n-r$ by $n-r$ matrix. It may be shown* that the matrix H defined by

$$H = (I-Q)^{-1},$$

called the fundamental matrix of the absorbing chain, has elements n_{ij} such that n_{ij} is the mean number of times the process is in the j^{th} non-absorbing state before absorption occurs, given that it started in the i^{th} state. The sum

$$T_i = \sum_j n_{ij}$$

*See Appendix B.

is then the mean number of times in any non-absorbing state for the i^{th} starting state and may therefore be related to the system MTF associated with the i^{th} initial state.

To illustrate the functional dependence of the MTF on the failure rate λ and the repair rate μ , the system MTF is derived parametrically in Appendix B for $N=5$, $M \geq 3$ to be

$$\text{MTF} = \frac{134\lambda^3 + 113\lambda^2\mu + 37\lambda\mu^2 + 6\mu^3}{60\lambda^3[2\lambda + \mu]}$$

when the system starts with all 5 machines initially operating.

The MTF for a number of other cases were produced by a computer program which numerically inverted the matrix H using $1/\lambda=70$ hours for the RTCC mean time to failure experience and $1/\mu=2$ hours for the mean time to repair experience. These results are tabulated in Table 5. Notice that the largest increment in reliability is between having one and two repair crews, as was true for the availability.

No. of Repair Crews M	N=5 MTF	N=6 MTF
1	4641	28,021
2	9506	114,037
3	9667	159,951
4 or more	9667	174,572

Mean Time to System Failure (hours)

Table 5

The effect of the initial state on the MTF is also available from H as noted above. Table 6 shows the influence of the initial state to be small for the particular failure and repair time statistics used.

Initial No. of Working Machines	N=5, M _≥ 3	N=6, M _≥ 4
	MTF	MTF
6		174,572
5	9667	174,560
4	9653	174,464
3	9513	172,773
2	9338	170,340

Mean Time to System Failure (hours)

Table 6

B. System Reliability

The reliability of the RTCC with respect to supporting two critical mission phases, as modeled in Figure 4, is derived in Appendix C to be:

$$\begin{aligned}
 R(t) = & 1.0003e^{-t/9667 \text{ hrs}} - .3843 \cdot 10^{-3}e^{-t/1.93 \text{ hrs}} \\
 & + .8806 \cdot 10^{-4}e^{-t/.936 \text{ hrs}} + .9939 \cdot 10^{-6}e^{-t/.621 \text{ hrs}}
 \end{aligned}$$

for N=5, M=3.

This function is tabulated below at some selected values of t, along with the values of $R^T(t)$, which is R(t) truncated to just the dominant first term, and the approximation $R(t) = e^{-t/MTF}$.

t(hours)	R(t)	$R^T(t)$	$e^{-t/MTF}$
1	.999993	1.000191	.999896
5	.999749	.999777	.999483
10	.999258	.999260	.998966
20	.998227	.998227	.997933
30	.997195	.997195	.996900
40	.996164	.996164	.995871
50	.995134	.995134	.994842
100	.990000	.990000	.989711
1000	.901989	.901989	.901750
9700	.366723	.366723	.366724

System Reliability $R(t)$ and Two
Approximations for $R(t)$

Table 7

Note that the approximation $R(t) \sim e^{-t/MTF}$ is conservative out to the system mean time to failure [i.e., lies below the true reliability function $R(t)$]. This aspect is elaborated on in Appendix C.

Figure 5 shows the system reliability $R(t)$ associated with the model of Figure 4 with the total number of machines N varying from $N=2$ to $N=5$. The explicit expressions for the reliability functions in these cases are given in Appendix C.


IV. SUMMARY

The RTCC has been modeled as a system of identical parallel redundant elements, having exponential failure and repair distributions. General results are obtained for the availability of an arbitrary number (K) out of the total number (N) of system elements with an arbitrary number (M) of repair crews. In addition, general results are obtained for the system availability when a worst case requirement of the simultaneous support of two critical mission phases is assumed. With this system requirement, a general technique for computing the system mean time to failure is illustrated, and the explicit system reliability function for several cases of interest is found.

Using the system parameters characteristic of the RTCC, with a total of five machines and at least five repair crews the model predicts that all five machines will be operating greater than 86% of the time and four or more machines will operate greater than 99% of the time. In addition, on a system basis, for five machines and three or more repair crews the model predicts:

1. A system availability greater than 99.99%;
2. A mean time to system failure of 9667 hours; and
3. A fifty hour reliability greater than .9951.

The principal deficiency of the model lies in the absence of consideration of periodic maintenance. However, since this activity is subject to schedule, and since switch-over to an active role will be rapid if required, the effect of periodic maintenance on the results obtained here should be small.



J. J. Rocchio

1031-JJR-jdc

Attachments
Appendices A, B and C
Figures 1-5

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APPENDIX A

System Availability - The Markov Model With Reflecting Barriers

To calculate system availability or the probability of having at least a given number of machines working, the system is modeled as a Markov chain with reflecting barriers as shown in Figure 1. A Markov chain of this type is characterized by its probability transition matrix P where p_{ij} is the probability of going from state S_i to state S_j . The matrix P applicable to the system with N machines and N or more repair crews may be written by inspection from Figure 1 as:

$$P = \begin{bmatrix} 1-N\mu\Delta t & N\mu\Delta t & 0 & \dots & 0 \\ \lambda\Delta t & 1-[(N-1)\mu + \lambda]\Delta t & (N-1)\mu\Delta t & \dots & 0 \\ 0 & 2\lambda\Delta t & 1-[(N-2)\mu + 2\lambda]\Delta t & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 2\mu\Delta t & 0 \\ 0 & 0 & 0 & \dots & (N-1)\lambda\Delta t & 1-[\mu+(N-1)\lambda]\Delta t & \mu\Delta t \\ 0 & 0 & 0 & \dots & 0 & N\lambda\Delta t & 1-N\lambda\Delta t \end{bmatrix}$$

Let the row vector $p^{(0)}$ characterize the initial state probabilities, i.e., $p_i^{(0)}$ is the probability that the system starts in state S_i . It may be shown that:

$$p^{(1)} = p^{(0)}P \quad ; \quad (A-1)$$

and in general that:

$$p^{(n)} = p^{(n-1)}P \quad . \quad (A-2)$$

Appendix A (contd.)

Using (A-1) recursively it is clear that (A-2) may be rewritten:

$$\underline{p}^{(n)} = \underline{p}^{(0)} P^n ; \quad (A-3)$$

which says that the probabilities of being in each state after n steps of the process are given by the product of the initial probability vector and the n^{th} power of the probability transition matrix.

Now consider a probability vector $\underline{\omega}$ such that

$$\underline{\omega} P = \underline{\omega} \quad (A-4)$$

and assume that $\underline{\omega}$ was chosen for $\underline{p}^{(0)}$; then

$$\underline{p}^{(n)} = \underline{\omega} P^n = \underline{\omega} = \underline{p}^{(0)} ,$$

or the probability of being in any given state remains constant at all steps of the process. In this case the process is said to be in equilibrium and the vector $\underline{\omega}$ is called the fixed point of the matrix P . The components of $\underline{\omega}$ give the equilibrium probabilities of state occupancy for the process and these are what is desired for computation of system availability. It can be shown* for example that under certain conditions

$$\lim_{n \rightarrow \infty} \underline{p}^{(n)} = \underline{p}^{(0)} P^n = \underline{\omega}$$

for any $\underline{p}^{(0)}$; therefore after a large number of steps the probability of the process being in state S_j is close to ω_j no matter what the starting state of the system is.

From (A-4) the fixed point probabilities of the system satisfy:

$$\underline{\omega} \cdot [P - I] = 0 \quad (A-5)$$

*See Finite Mathematics with Business Application, J. G. Kemeny et al., Prentice Hall, 1962.

Appendix A (contd.)

and since $\underline{\omega}$ is a probability vector

$$\sum_{i=0}^N \omega_i = 1 \quad (A-6)$$

If the system of equations (A-5) is solvable, for other than the trivial case $\underline{\omega} = 0$, it has an infinite number of solutions, only one of which will also satisfy (A-6). From the matrix P and (A-5) we may write:

$$\begin{aligned} N\mu\omega_0 - \lambda\omega_1 &= 0 \\ -N\mu\omega_0 + [(N-1)\mu + \lambda]\omega_1 - 2\lambda\omega_2 &= 0 \\ \vdots \\ -2\mu\omega_{N-2} + [\mu + (N-1)\lambda]\omega_{N-1} - N\lambda\omega_N &= 0 \\ -\mu\omega_{N-1} + N\lambda\omega_N &= 0 \end{aligned}$$

These equations may be manipulated into the following form starting from the last one:

$$\omega_{N-1} = \frac{N\lambda}{\mu} \omega_N$$

Adding the last two equations gives:

$$\omega_{N-2} = \frac{N-1}{2} \frac{\lambda}{\mu} \omega_{N-1} = \frac{(N-1)N}{2 \cdot 1} \left(\frac{\lambda}{\mu} \right)^2 \omega_N$$

and repeating this process gives in general:

$$\omega_{N-K} = \frac{N-K+1}{K} \frac{\lambda}{\mu} \omega_{N-K+1} = \binom{N}{N-K} \left(\frac{\lambda}{\mu} \right)^{N-K} \omega_N$$

Appendix A (contd.)

Substituting these results into (A-6) gives:

$$\omega_N \sum_{K=0}^N \binom{N}{K} \left(\frac{\lambda}{\mu}\right)^K = 1$$

or

$$\omega_N = \frac{1}{\left(1 + \frac{\lambda}{\mu}\right)^N}.$$

Thus the desired result is:

$$\omega_K = \frac{\binom{N}{K} \left(\frac{\lambda}{\mu}\right)^{N-K}}{\left(1 + \frac{\lambda}{\mu}\right)^N} = \frac{\binom{N}{K} \left(\frac{\mu}{\lambda}\right)^K}{\left(1 + \frac{\mu}{\lambda}\right)^N}. \quad (A-7)$$

The effect on the equilibrium state occupancy probabilities ω due to the number of available repair crews may be obtained from a similar argument. Let M , $1 \leq M < N$ be the number of repair crews available to the system. We must consider the effect on the transition matrix P which we now denote by $P(M)$. If we denote a row of P by \underline{P}_j , then it is clear that

$$\underline{P}_j = \underline{P}(M)_j$$

for

$$j \geq N-M$$

which corresponds to the fact that when M or less machines are failed, the transition probabilities are the same whether we have M , N or more repair crews since the surplus crews must be idle. Now for $j < N-M$ a transition probability of the form:

$$P_{j,j+1} = (N-j)\mu\Delta t$$

Appendix A (contd.)

becomes

$$p^{(M)}_{j,j+1} = M\mu\Delta t \quad ;$$

and

$$p_{j,j} = 1 - [(N-j)\mu\Delta t + j\lambda\Delta t]$$

becomes

$$p^{(M)}_{j,j} = 1 - [M\mu\Delta t + j\lambda\Delta t] \quad .$$

With these observations the set of equations:

$$\underline{\omega}[P(M)-I] = 0$$

may be written as:

$$M\mu\omega_0 - \lambda\omega_1 = 0$$

$$-M\mu\omega_0 + [M\mu + \lambda]\omega_1 - 2\lambda\omega_2 = 0$$

$$\vdots$$

$$-M\mu\omega_{N-M-1} + [M\mu + (N-M)\lambda]\omega_{N-M} - (N-M+1)\lambda\omega_{N-M+1} = 0$$

$$-M\mu\omega_{N-M} + [(M-1)\mu + [N-(M-1)\lambda]]\omega_{N-M+1} - (N-M+2)\lambda\omega_{N-M+2} = 0$$

$$\vdots$$

$$-2\mu\omega_{N-2} + [\mu + (N-1)\lambda]\omega_{N-1} - N\lambda\omega_N = 0$$

$$-\mu\omega_{N-1} + N\lambda\omega_N = 0 \quad .$$

Appendix A (contd.)

Proceeding as before we get

$$\omega_{N-1} = \frac{N\lambda}{\mu} \omega_N$$

$$\omega_{N-2} = \frac{(N-1)}{2} \frac{\lambda}{\mu} \omega_{N-1} = \binom{N}{N-2} \left(\frac{\lambda}{\mu}\right)^2 \omega_N$$

$$\vdots$$

$$\omega_{N-M} = \binom{N}{M} \left(\frac{\lambda}{\mu}\right)^M \omega_N$$

$$\omega_{N-M-1} = \frac{N-M}{M} \left(\frac{\lambda}{\mu}\right) \omega_{N-M} = \frac{N-M}{M} \binom{N}{M} \left(\frac{\lambda}{\mu}\right)^{M+1} \omega_N$$

or in general

$$\omega_K = \binom{N}{K} \left(\frac{\lambda}{\mu}\right)^{N-K} \omega_K, \quad \text{for } N-M \leq K \leq N;$$

and

$$\omega_K = \frac{(N-M)!}{M^{N-M-K} K!} \binom{N}{M} \left(\frac{\lambda}{\mu}\right)^{N-K} \omega_N, \quad \text{for } 0 \leq K < N-M.$$

Therefore

$$\omega_N \left[\sum_{K=N-M}^N \binom{N}{K} \left(\frac{\lambda}{\mu}\right)^{N-K} + \sum_{K=0}^{N-M-1} \binom{N}{M} \frac{(N-M)!}{M^{N-M-K} K!} \left(\frac{\lambda}{\mu}\right)^{N-K} \right] = 1.$$

Hence, the desired result is

$$\omega_K = \frac{\binom{N}{K} \left(\frac{\lambda}{\mu}\right)^{N-K}}{\sigma} \quad \text{for} \quad N-M \leq K \leq N,$$

Appendix A (contd.)

and

$$\omega_K = \frac{\frac{(N-M)!}{K! M^{N-M-K}} \left(\frac{N}{M}\right) \left(\frac{\lambda}{\mu}\right)^{N-K}}{\sigma} \quad \text{for } 0 \leq K < N-M,$$

where

$$\sigma = \sum_{K=N-M}^N \left(\frac{N}{K}\right) \left(\frac{\lambda}{\mu}\right)^{N-K} + \sum_{K=0}^{N-M-1} \frac{(N-M)!}{K! M^{N-M-K}} \left(\frac{N}{M}\right) \left(\frac{\lambda}{\mu}\right)^{N-K}.$$

Having obtained the equilibrium probabilities of state occupancy for the general model with an arbitrary number of repair crews, we are in a position to investigate the models of Figures 2 and 3 which are applicable to the RTCC.

The method of solution in these cases proceeds exactly as above, however the splitting of state S_2 into two states S_{21} and S_{22} as shown in the state diagrams adds an additional row and column to the transition matrix. Previously we had:

$$p_{2,1} = 2\lambda\Delta t$$

$$p_{2,2} = 1 - [(N-2)\mu + 2\lambda]\Delta t$$

$$p_{2,3} = (N-2)\mu\Delta t$$

In the transition matrix applicable to the system of Figure 2, P' we have

$$p'_{21,1} = 2\lambda\Delta t$$

$$p'_{21,21} = 1 - [(N-2)\mu + 2\lambda]\Delta t$$

Appendix A (contd.)

$$p'_{21,3} = (N-2)\mu\Delta t$$

$$p'_{22,1} = 2\lambda\Delta t$$

$$p'_{22,22} = 1 - [(N-2)\mu + 2\lambda]\Delta t$$

$$p'_{22,3} = (N-2)\mu\Delta t$$

In addition, whereas

$$p_{3,2} = 3\lambda\Delta t$$

we now have

$$p'_{3,21} = 2\lambda\Delta t$$

and

$$p'_{3,22} = \lambda\Delta t$$

From the system of equations

$$\omega \cdot [P' - I] = 0$$

(A-8)

Appendix A (contd.)

it can be shown that

$$[(N-2)\mu+2\lambda]\omega_{22} - \lambda\omega_3 = 0$$

or

$$\omega_{22} = \frac{\lambda\omega_3}{(N-2)\mu+2\lambda} \quad (A-9)$$

Also by examination of the set of equations of (A-8) compared with those of (A-5) we can show that the substitution

$$\omega_{22} + \omega_{21} = \omega_2$$

reduces (A-8) to (A-5), therefore leaving the equilibrium probabilities for states other than S_2 unchanged. Thus the previous solution (A-7), plus equations (A-9) and (A-10) provide the equilibrium probabilities for the model of Figure 2.

In the general case with M repair crews (A-9) becomes

$$\omega_{22} = \frac{\lambda\omega_3}{\alpha\mu+2\lambda}$$

where

$$\alpha = \min[N-2, M]$$

The results applicable to the model of Figure 3 are very similar to the above. The governing equation for ω_{22} in this case becomes

Appendix A (contd.)

$$[\alpha\mu+2\lambda+\delta]\omega_{22} - \lambda\omega_3 = 0$$

or

$$\omega_{22} = \frac{\lambda\omega_3}{\alpha\mu+2\lambda+\delta},$$

with all other relations unchanged.

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APPENDIX B

System Reliability - The Markov Model With Absorbing Barriers

Consideration of the Markov chain of Figure 4 allows computation of the mean time to system failure. In this model the system down states S_1 and S_{22} are made absorbing barriers and we are interested in the mean time to absorption. Due to algebraic complexity, closed form solutions for the general case were not derived. A parametric solution for one case will be derived below. The results presented in the body of the memorandum were computed by numerical techniques and involve the inversion of a relatively sparse matrix.

In general, the probability transition matrix of an absorbing Markov chain may be written in the standard form:

$$P = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix} \quad (B-1)$$

if the states are ordered with the absorbing states first. With P in the above form it is easily shown that P^n is of the form:

$$P^n = \begin{bmatrix} I & 0 \\ * & Q^n \end{bmatrix} \quad (B-2)$$

where * refers to an unspecified submatrix.

We are interested in the mean number of times (η_{ij}) the process is in state S_j , given it starts in state S_i , where i and j are nonabsorbing states. Since the probability of eventual absorption is one, we assume the means exist. Let $x_{ij}(k)$ be a random variable such that

$$x_{ij}(k) = \begin{cases} 1 & \text{if the process is in state} \\ & j \text{ at step } k \text{ given that it} \\ & \text{started in state } i; \\ 0 & \text{otherwise.} \end{cases}$$

Appendix B (contd.)

Let

$$s_{ij} = \sum_{k=0}^{\infty} x_{ij}(k) ,$$

i.e., s_{ij} is the total number of times the process is in state S_j given that it started in state S_i .

Now

$$n_{ij} = \overline{s_{ij}} = \overline{\sum_{k=0}^{\infty} x_{ij}(k)} = \sum_{k=0}^{\infty} \overline{x_{ij}(k)} , \quad (B-3)$$

but

$$\overline{x_{ij}(k)} = \{Q^k\}_{ij} ,$$

where Q is defined by (B-1).

Therefore, writing (B-3) in matrix form we have:

$$H = [n_{ij}] = I + Q + Q^2 + \dots = (I-Q)^{-1} .$$

Hence, the elements of $(I-Q)^{-1}$ are the mean number of steps in each nonabsorbing state for each possible nonabsorbing starting state. Thus the row sums of H

$$T_i = \sum_j n_{ij}$$

give the mean number of steps to absorption for starting state S_i .

An example will illustrate the computational procedure. The transition matrix P for the Markov chain of Figure 4 for five machines ($N=5$) is:

Appendix B (contd.)

States	1	22	21	3	4	5
1	1	0	0	0	0	0
22	0	1	0	0	0	0
P = 21	$2\lambda\Delta t$	0	$1-(2\lambda+3\mu)\Delta t$	$3\mu\Delta t$	0	0
3	0	$\lambda\Delta t$	$2\lambda\Delta t$	$1-(3\lambda+2\mu)\Delta t$	$2\mu\Delta t$	0
4	0	0	0	$4\lambda\Delta t$	$1-(4\lambda+\mu)\Delta t$	$\mu\Delta t$
5	0	0	0	0	$5\lambda\Delta t$	$1-5\lambda\Delta t$

From P it is easily seen that

$$I-Q = \Delta t \begin{bmatrix} 2\lambda+3\mu & -3\mu & 0 & 0 \\ -2\lambda & 3\lambda+2\mu & -2\mu & 0 \\ 0 & -4\lambda & 4\lambda+\mu & -\mu \\ 0 & 0 & -5\lambda & 5\lambda \end{bmatrix}$$

After algebraic manipulation it can be shown that:

$$H=(I-Q)^{-1}=\frac{1}{\Delta t D} \begin{bmatrix} 60\lambda^3 & 60\lambda^2\mu & 30\lambda\mu^2 & 6\mu^3 \\ 40\lambda^3 & 20\lambda^2(2\lambda+3\mu) & 10\lambda(2\lambda\mu+3\mu^2) & 2\mu^2(2\lambda+3\mu) \\ 40\lambda^3 & 20\lambda^2(2\lambda+3\mu) & 5\lambda(6\lambda^2+7\lambda\mu+6\mu^2) & 6\lambda^2\mu+7\lambda\mu^2+6\mu^3 \\ 40\lambda^3 & 20\lambda^2(2\lambda+3\mu) & 5\lambda(6\lambda^2+7\lambda\mu+6\mu^2) & 24\lambda^3+18\lambda^2\mu+7\lambda\mu^2+6\mu^3 \end{bmatrix}$$

where

$$D = 60\lambda^3[2\lambda+\mu] \quad .$$

Appendix B (contd.)

Since each step of the Markov process take Δt units of time, τ_i , the mean time to system failure starting from state S_i , is given by

$$\tau_i = \Delta t T_i = \Delta t \sum_j n_{ij}$$

or for the case above

$$\tau_5 = \frac{134\lambda^3 + 113\lambda^2\mu + 37\lambda\mu^2 + 6\mu^3}{60\lambda^3[2\lambda + \mu]},$$

the mean time to system failure starting with all machines working.

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APPENDIX C

System Reliability - Solution of the Set of Differential Equations

Consider the system model illustrated by Figure 4. The system reliability $R(t)$ is the probability that the system operates satisfactorily for t units of time. Since this function will depend on the state of the system at $t=0$ we should write $R(t,j)$ as the reliability of the system given that at $t=0$ the system is in state S_j . Let the probability that the system is in state S_1 at time t , given that it started in state S_j , be $P_1(t,j)$. We may then write:

$$R(t,j) = 1 - \left[P_1(t,j) + P_{22}(t,j) \right] \quad . \quad (C-1)$$

Rather than carry the notation of the starting state, we will assume from now on that the system starts with all machines working. Thus C-1 becomes

$$R(t) = 1 - \left[P_1(t) + P_{22}(t) \right]$$

Since we are not interested in distinguishing the states S_1 and S_{22} , they may be combined into a single system failed state S_* for which

$$P_*(t) = P_1(t) + P_{22}(t)$$

and therefore

$$R(t) = 1 - P_*(t) \quad (C-2)$$

Appendix C (Contd.)

It may be shown* that the state probabilities $P_i(t)$ satisfy a set of simultaneous linear differential equations, which for the model of Figure 4 with five machines ($N=5$) and three or more repair crews ($M \geq 3$) are given below:

$$P_5'(t) = -5\lambda P_5(t) + \mu P_4(t)$$

$$P_4'(t) = 5\lambda P_5(t) - (4\lambda + \mu)P_4(t) + 2\mu P_3(t)$$

$$P_3'(t) = 4\lambda P_4(t) - (3\lambda + 2\mu)P_3(t) + 3\mu P_2(t)$$

$$P_2'(t) = 2\lambda P_3(t) - (2\lambda + 3\mu)P_2(t)$$

$$P_*'(t) = \lambda P_3(t) + 2\lambda P_2(t)$$

where $P_i'(t)$ denotes the time derivative of $P_i(t)$.

Taking Laplace transforms of these equations with $P_5(0) = 1$ as an initial condition yields the following relation in matrix form:

$$M(s) \cdot \underline{L}(s) = \delta \quad ;$$

with

$$M(s) = \begin{bmatrix} s+5\lambda & -\mu & 0 & 0 & 0 \\ -5\lambda & 4\lambda+\mu+s & -2\mu & 0 & 0 \\ 0 & -4\lambda & 3\lambda+2\mu+s & -3\mu & 0 \\ 0 & 0 & -2\lambda & 2\lambda+3\mu+s & 0 \\ 0 & 0 & -\lambda & -2\lambda & s \end{bmatrix} \quad ;$$

$$\underline{L}(s) = \begin{bmatrix} L_5(s) \\ L_4(s) \\ L_3(s) \\ L_2(s) \\ L_*(s) \end{bmatrix} \quad ; \quad \delta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad ; \text{ and } L_i(s) = L[P_i(t)] \quad .$$

*See for example "The Reliability of Some Simple Redundant, Repairable Systems," Bellcomm Technical Memorandum, TM-64-2131-1, January 28, 1964, by I. D. Nehama.

Appendix C (Contd.)

This system may be solved for the transforms $L_i(s)$ using standard matrix techniques. After considerable algebra one obtains:

$$L_*(s) = \frac{A(s)}{B(s)}$$

where

$$\begin{aligned} A(s) &= 20\lambda^3(s+6\lambda+3\mu) \\ B(s) &= s[s^4+(14\lambda+6\mu)s^3 + (71\lambda^2+49\lambda\mu+11\mu^2)s^2 \\ &\quad + (154\lambda^3+113\lambda^2\mu+37\lambda\mu^2+6\mu^3)s + 120\lambda^4+60\lambda^3\mu] \end{aligned}$$

Now equation C-2 may be written

$$R(t) = 1 - L^{-1}[L_*(s)]$$

The inverse transform of $L_*(s)$, i.e., $P_*(t)$ may be found by a partial fraction expansion of $L(s)$:

$$\begin{aligned} L_*(s) &= \frac{A(s)}{B(s)} = \frac{A(s)}{s(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \\ &= \frac{K_0}{s} + \frac{K_1}{(s-s_1)} + \frac{K_2}{(s-s_2)} + \frac{K_3}{(s-s_3)} + \frac{K_4}{(s-s_4)} \end{aligned}$$

As

$$K_0 = s \frac{A(s)}{B(s)} \Big|_{s=0} = 1$$

we may write $R(t)$ directly (assuming real, distinct roots for $B(s)$) as:

$$R(t) = - \left(K_1 e^{s_1 t} + K_2 e^{s_2 t} + K_3 e^{s_3 t} + K_4 e^{s_4 t} \right) \quad (C-3)$$

As the roots of $B(s)$ are very difficult to obtain algebraically, they were obtained numerically using the values characteristic of the RTCC namely, a mean time to failure ($1/\lambda$) of 70 hours and a mean time to repair ($1/\mu$) of 2 hours. The solution yields the parameters of equation C-3 as:

$$\begin{aligned}
K_1 &= -1.000295 & ; & \quad s_1 = -0.103448 \cdot 10^{-3} \text{ hrs}^{-1} \\
K_2 &= 0.384316 \cdot 10^{-3} & ; & \quad s_2 = -0.518470 \text{ hrs}^{-1} \\
K_3 &= -0.880612 \cdot 10^{-4} & ; & \quad s_3 = -1.069435 \text{ hrs}^{-1} \\
K_4 &= -0.993931 \cdot 10^{-6} & ; & \quad s_4 = -1.611992 \text{ hrs}^{-1}
\end{aligned}$$

Let the reliability function for the model of Figure 4 for arbitrary N be denoted as $R(t, N)$. If the roots of the characteristic equation

$$| M(s) | = 0$$

are real, negative and distinct we may write

$$R(t, N) = \sum_{i=1}^{N-1} K_i e^{s_i t} \quad (C-4)$$

Solutions for $R(t, N)$ for $N = 2, 3$, and 4 have been found in addition to the solution for $N = 5$ given above. For $N = 2$, virtually by inspection,

$$R(t, 2) = e^{-2\lambda t}.$$

For $N = 3$ and 4 , the parameters applicable to Equation (C-4) are given in Table C1 below.

	$N = 3$	$N = 4$
K_1	1.0014	1.0028
K_2	$-0.140 \cdot 10^{-2}$	$-0.278 \cdot 10^{-2}$
K_3		$0.193 \cdot 10^{-5}$
s_1	$-0.1504 \cdot 10^{-1} \text{ hrs}^{-1}$	$-0.1506 \cdot 10^{-2} \text{ hrs}^{-1}$
s_2	-0.5564 hrs^{-1}	-0.5428 hrs^{-1}
s_3		-1.0842 hrs^{-1}

Parameters of $R(t, N)$ (Eq. C-4) for $N = 3$ and 4 ;
 $1/\lambda = 70$ failures/hour, and $1/\mu = 2$ repairs/hour

Table C-1

From the structure of $M(s)$ we observe

$$\lim_{\lambda \rightarrow 0} |M(s)| = s^2 (s + \mu) (s + 2\mu) \dots (s + (N-2)\mu)$$

which suggests that if the spectral sequence of Equation C-4 ($\{s_i\}$) is placed in numerically decending order, i.e.,

$$\{s_i\} = s_1, > s_2, > \dots > s_{N-1}$$

it will be asymptotic to the sequence

$$0, -\mu, -2\mu, \dots, -(N-2)\mu$$

as $\lambda \rightarrow 0$. From this result, and the cases considered above we conjecture that in general (for $\lambda \ll \mu$), $R(t, N)$ is dominated by the first term of (C-4), and may be approximated by

$$R(t, N) \simeq (1 + \delta) e^{s_1(N)t}, \quad \delta \ll 1 \quad (C-5)$$

and therefore that

$$s_1(N) \simeq 1/\text{MTF}(N) \quad (C-6)$$

This suggests that a useful approximation to $R(t, N)$ for $\lambda \ll \mu$ and any N is

$$R^*(t, N) = e^{-t/\text{MTF}(N)} \quad (C-7)$$

i.e., the system behaves as though it consisted of a single element with the appropriate MTF. Moreover, it can be shown from the form of the Laplace transform of $R(t, N)$ that:

$$R(t, N) \big|_{t=0+} > R^*(t, N) \big|_{t=0+} \text{ for } N \geq 4.$$

Also from (C-5) and (C-6) it can be shown that

$$R(t, N) = R^*(t, N) \text{ at } t \simeq \text{MTF}.$$

Appendix C (Contd.)

Thus, with the assumptions made, $R^*(t, N)$ provides a lower bound to the system reliability over the range $0 \leq t < \sqrt{MTF}$, and requires only computation of the system MTF .

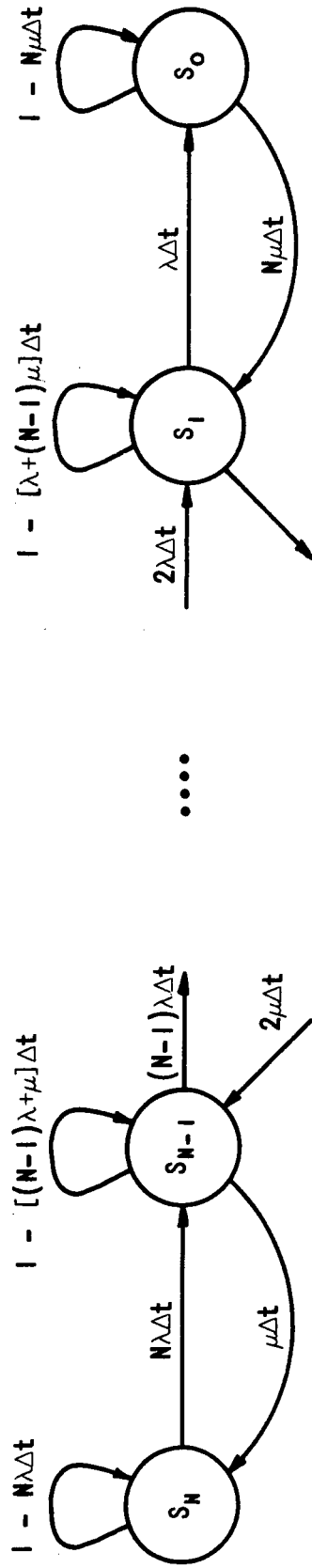


FIGURE 1 - MARKOV CHAIN - N MACHINES AND N REPAIR CREWS

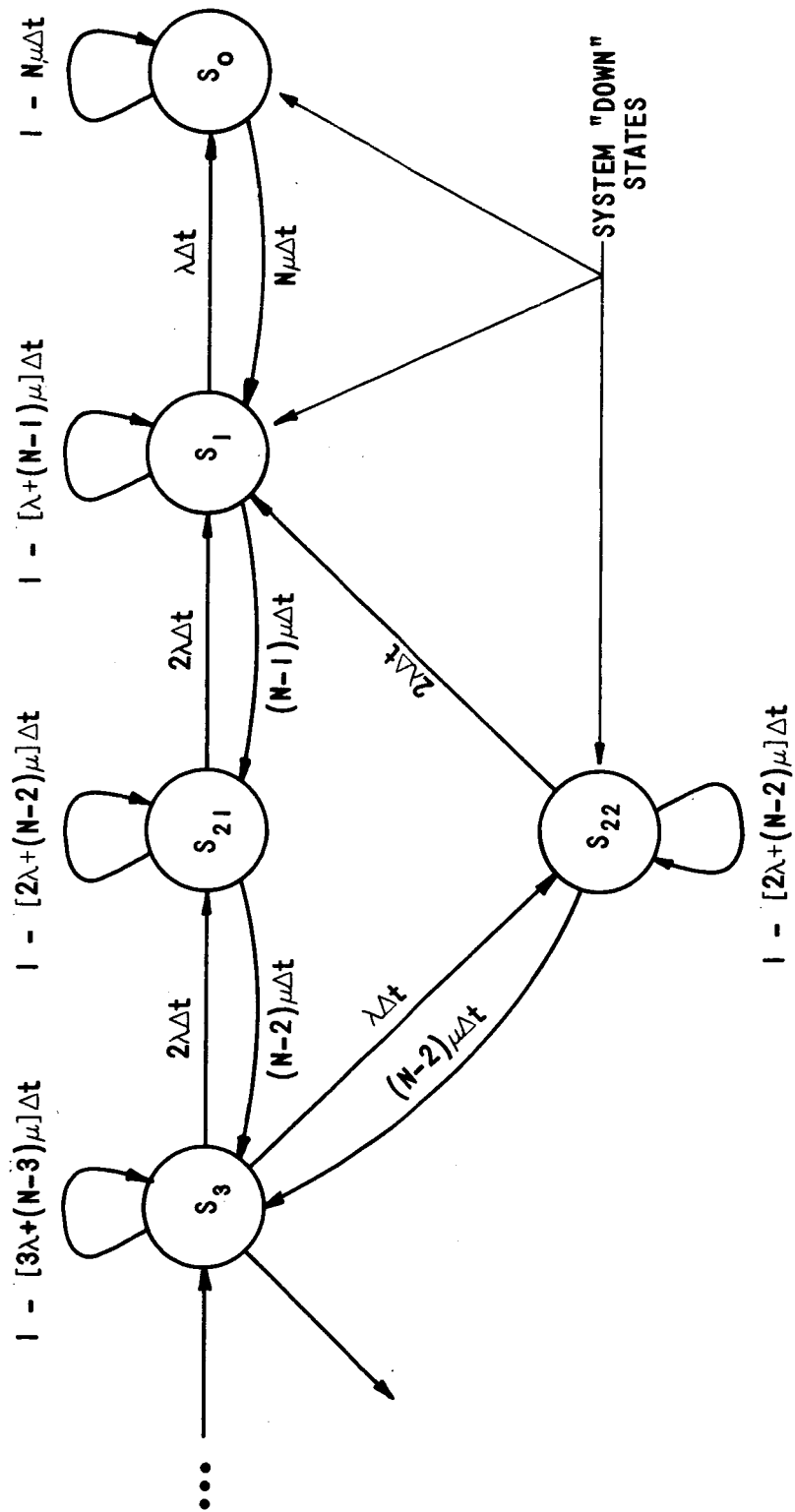


FIGURE 2 - MARKOV CHAIN MODEL FOR RTCC SUPPORTING TWO CRITICAL MISSION PHASES

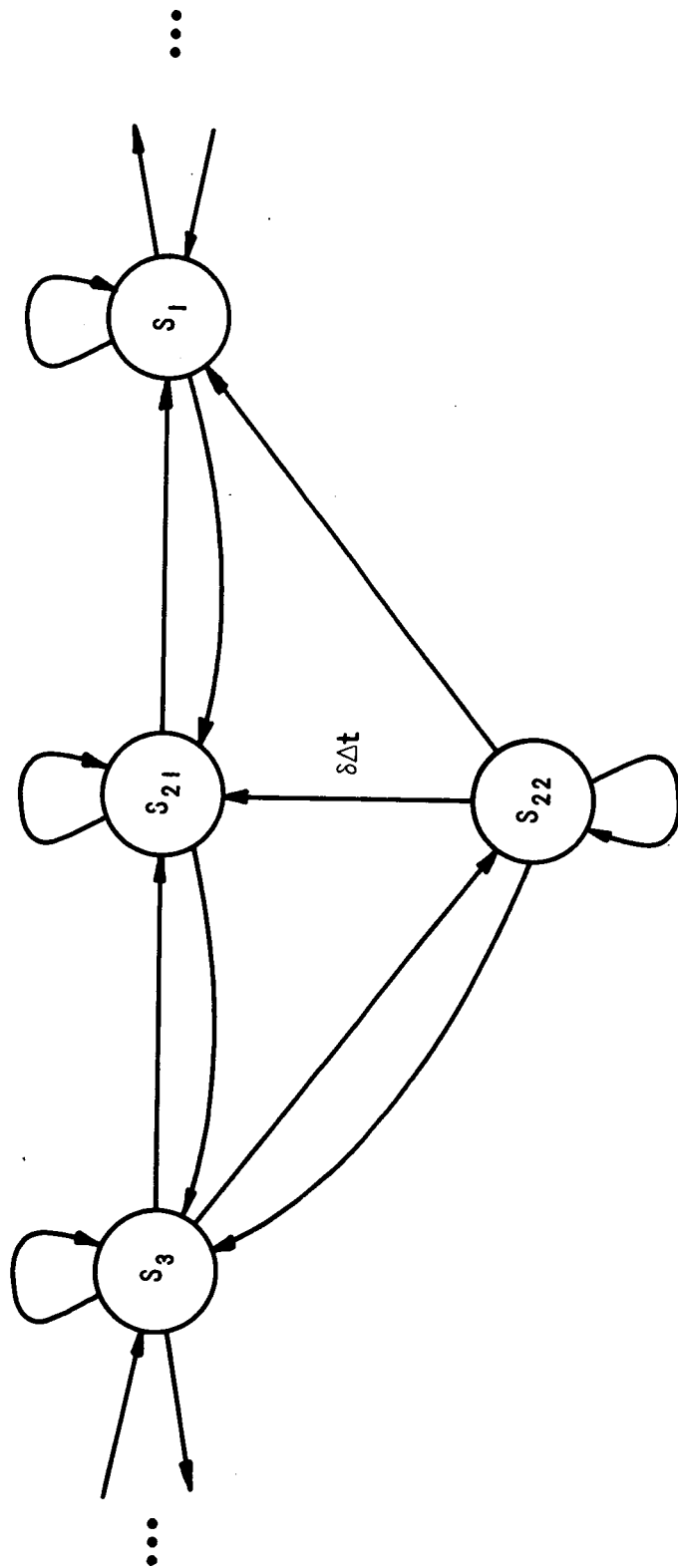


FIGURE 3 - ALTERNATIVE RTCC MODEL WITH SWITCHOVER WHERE THE SYSTEM HAS TWO MACHINES SUPPORTING ONE OF TWO CRITICAL MISSION PHASES

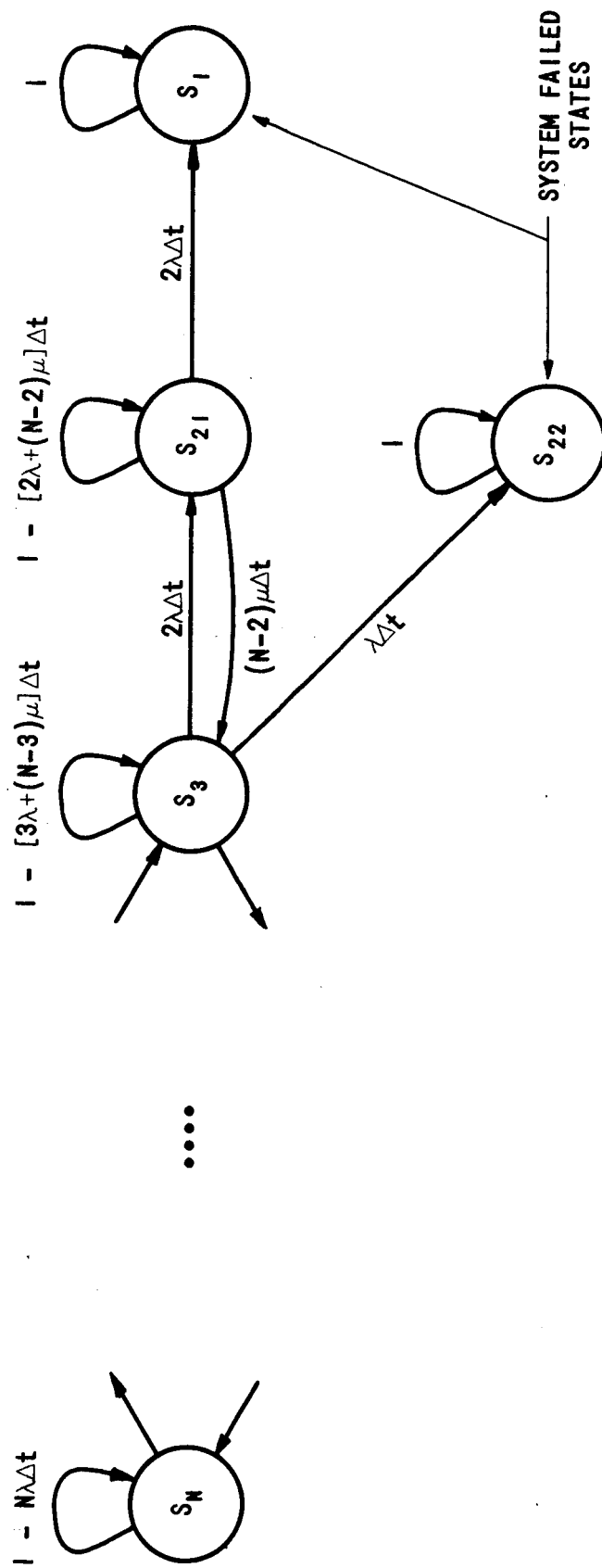


FIGURE 4 - MARKOV CHAIN WITH ABSORBING BARRIERS - SYSTEM MODEL FOR RELIABILITY COMPUTATIONS

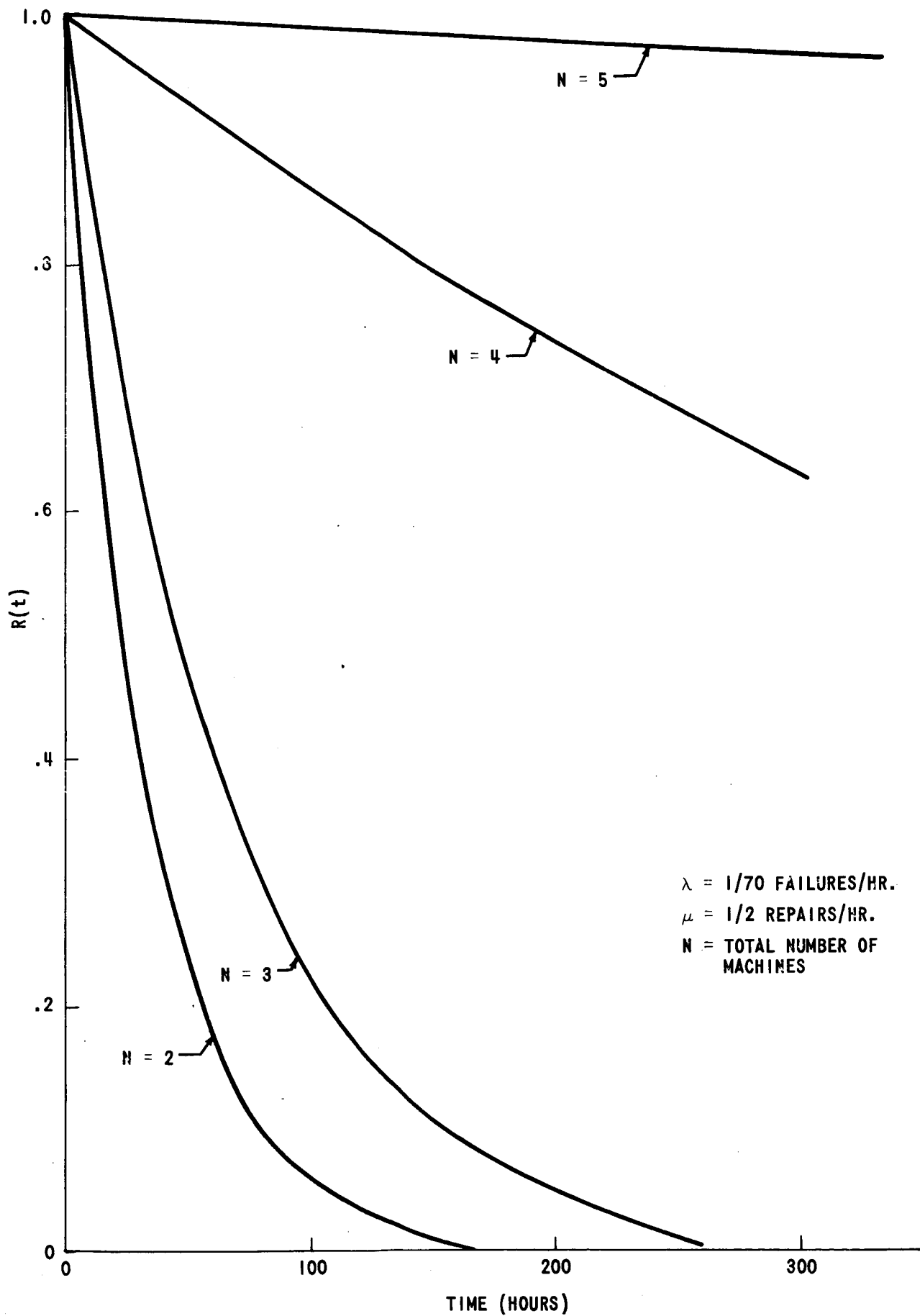


FIGURE 5 - RTCC RELIABILITY ($R(t)$) FOR SIMULTANEOUS SUPPORT OF TWO CRITICAL MISSION PHASES